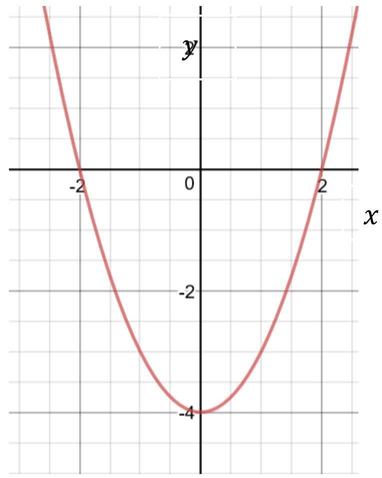
UNIT 6: RELATIONS AND FUNCTIONS

RELATIONS

A relation describes the relationship between two quantities, described:

- in words..................... a number is 4 less than the square of another number
- as an equation .... $y = x^2 - 4$
- a graph ....



The domain of a relation is the set of all the $\boldsymbol{x}$ -coordinates from the relation.

For the graph above: **domain =** $\{\boldsymbol{x}|\boldsymbol{x} \in \boldsymbol{R}\}$
(x is the set of all real numbers)

The range of a relation is the set of all the y-coordinates from the relation.

For the graph above: **range =** $\{\boldsymbol{y} \mid \boldsymbol{y} \geq -\boldsymbol{4}, \boldsymbol{y} \in \boldsymbol{R}\}$
(y is the set of all real numbers greater than and including $-\boldsymbol{4}$)

FUNCTIONS

A function is a special type of relation.

- In a function, every x-value is paired with just a single y -value.
- A relation is not a function if there is an x -value that is paired with more than one y-value.

FUNCTION NOTATION

$f(x)$ is" f of x " or "the value of function f for any given value of the variable x ".

- f is the name of this function; other letters or word can be used for names.
- x is the variable that f depends on.

Example: The function f is defined as $f(x) = x^2 - 4$

Calculate:

a) $f(3) = 3^2 - 4 = 9 - 4 = 5$

b) $f(-2) = (-2)^2 - 4 = 4 - 4 = 0$

c) $f(0) = 0^2 - 4 = 0 - 4 = -4$

d) $f(-1) =$

e) $f\left(\frac{2}{3}\right) =$

f) $f(4) =$

The graph of $f(x) = x - 4$. would have the points
$(3,5)$, $(-2,0)$, $(0,-4)$ etc.

QUADRATIC RELATIONS

The **degree** of a term is the number of its variable factors.

Example:

The degree of $3x^4$ is 4 because this term has 4 x- factors.

The degree of $-x^4y^2z$ is 7; it has 4 x -factors, 2 y-factors, 1 z-factor.

The degree of 4^3 is 0; it has no variable factors.

The degree of an equation with more than one term is the largest of the degrees from the equation's terms.

Example:

The degree *of* $3x^4 - x^4yz^2 = 64$ is 7 because it is largest of degrees 4, 7 and 0.

A ***Quadratic Relation*** is described by an equation of **degree 2**.

Example:

$y = 3x^2 - 2x + 1$ is a quadratic relation because its degree is 2. This relation is also a ***quadratic function.***

DEFINITION OF A QUADRATIC FUNCTION

The equation of any quadratic function can be written in the form:

$y = ax^2 + bx + c$ (or as $f(x) = ax^2 + bx + c$ in function notation)

where a, b and c are real numbers and $a \neq 0$

This form is called the **General Form** of a quadratic function.

The graph of any quadratic function is a ***parabola***.

The ***axis of symmetry*** is the line that "cuts" the parabola in half so that the part on one side of the line is a "mirror" image of the other side. This type of symmetry is called *reflective symmetry.*

The ***vertex of a parabola*** is the point between the two halves of the parabola; the point on the parabola that is also on the axis of symmetry.

On one side of the vertex, the parabola is falling; on the other side the parabola is rising. The vertex is the ***turning point*** of the parabola.

In a parabola that opens upward, the vertex is the lowest point on the parabola; it's the point on the parabola with the lowest y −coordinate. That y −coordinate is the ***minimum value*** of the quadratic function. It is needed to determine the range. In a parabola that opens downward, they-coordinate of the vertex is the ***maximum value*** of the quadratic function.

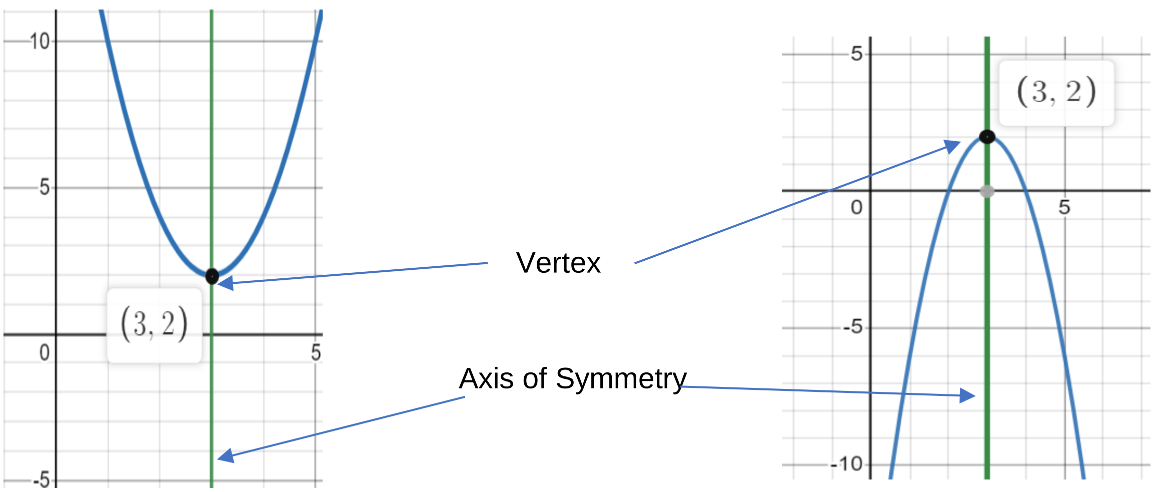
The direction of opening (upwards or downwards) is determined by the value of a in $y = ax^2 + bx + c$. If $a > 0$ the parabola opens upwards and if $a < 0$ the parabola opens downwards.

The ***domain*** of a relation is the set of all the x-coordinates from the relation. The ***range*** of a relation is the set of all the y-coordinates from the relation.

Example

The graphs of $y = 2x^2 - 12x + 20$ and $y = -2x^2 + 12x - 16$ are shown below.

$y = 2x^2 - 12x + 20$ $\qquad\qquad$ $y = -2x^2 + 12x - 16$



Vertex: (3,2)

Axis of Symmetry: $x = 3$

Direction of opening: Up

Minimum or Maximum: Min

Min/Max Value: 2

Domain: $\{x|x \in R\}$ or just $x \in R$

Range: $\{\, y \mid y \geq 3, y \in R\}$
or just $y \geq 3, y \in R$

Vertex: (3,2)

Axis of Symmetry: $x = 3$

Direction of opening: Down

Minimum or Maximum: Max

Min/Max Value: 2

Domain: $\{x|x \in R\}$ or just $x \in R$

Range: $\{\, y \mid y \leq 2, y \in R\}$
or just $y \leq 2, y \in R$

GRAPHING QUADRATIC FUNCTIONS IN GENERAL FORM

To graph a parabola given in the form $y = ax^2 + bx + c$ we use the following steps:

Step 1: Find the axis of symmetry. Use the formula $\boldsymbol{x} = \frac{-\boldsymbol{b}}{\boldsymbol{2a}}$

Step 2: Find the coordinates of the vertex

Step 3: Find the coordinates of at least two points to the left and two points to the right of the vertex

Example

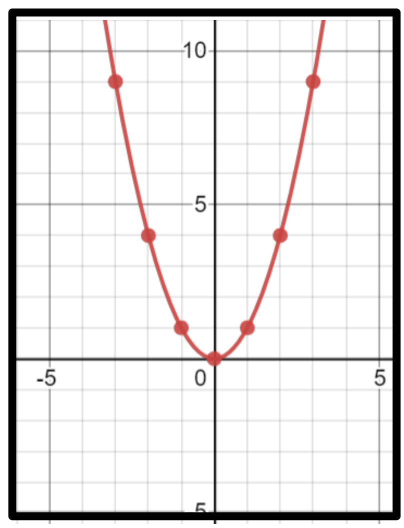
The *basic quadratic function* is described by equation $y = x^2$

Using a *table of values* draw the graph of $y = x^2$

Solution:

(± is "plus or minus"; ±1 means +1 or -1)

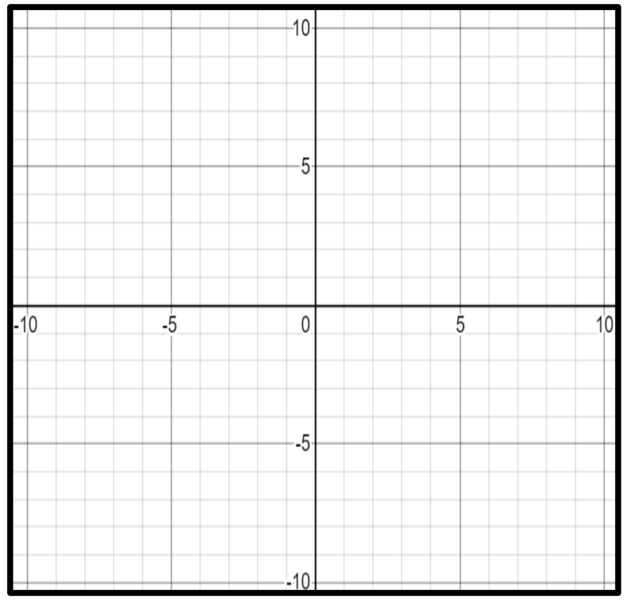
$\boldsymbol{x}$	$\boldsymbol{x^2 = y}$	$(\boldsymbol{x}, \boldsymbol{y})$
± 1	$(+1)^2 = 1$ $(-1)^2 = 1$	$(1,1)$ $(-1,1)$
± 2	$(+2)^2 = 4$ $(-2)^2 = 4$	$(2,4)$ $(-2,4)$
± 3	$(+3)^2 = 9$ $(-3)^2 = 9$	$(3,9)$ $(-3,9)$
0	$(0)^2 = 0$	$(0,0)$



Example: Graph the function $y = -2x^2 + 4x + 6$

Solution:

x	$y = -2x^2 + 4x + 6$	(x, y)



a) What is the equation for the axis of symmetry?

b) What are the coordinates of the vertex?

c) Does this function have a maximum or minimum value? What is this value?

d) What is the domain of this function?

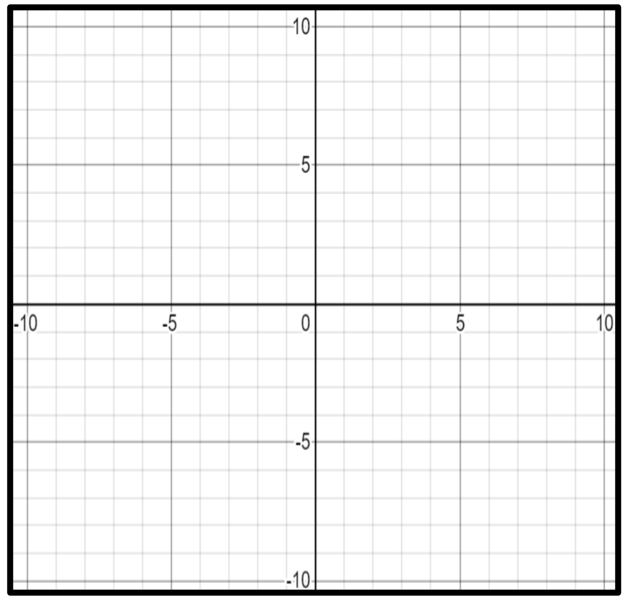
e) What is the range of this function?

f) What is the $y-$ intercept of this function?

Example: Graph the function $y = 3x^2 + 12x + 1$

Solution:

x	$y = 3x^2 + 12x + 1$	(x, y)



a) What is the equation for the axis of symmetry?

b) What are the coordinates of the vertex?

c) Does this function have a maximum or minimum value? What is this value?

d) What is the domain of this function?

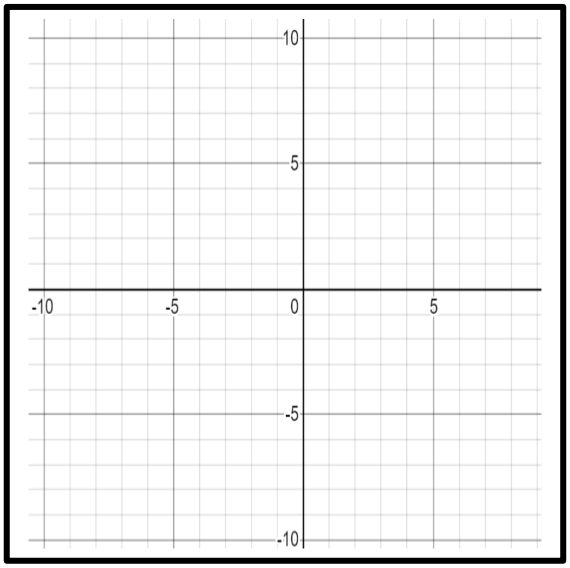
e) What is the range of this function?

f) What is the $y -$ intercept of this function?

PRACTICE PROBLEMS

1. Draw the following graphs. Plot the vertex and at least four other points on the graph.

a) $y = x^2 - 4x$



x	$y = x^2 - 4x$	(x, y)

y-intercept	
Max or Min? Value?	
Axis of symmetry	
Vertex	
Domain	
Range	

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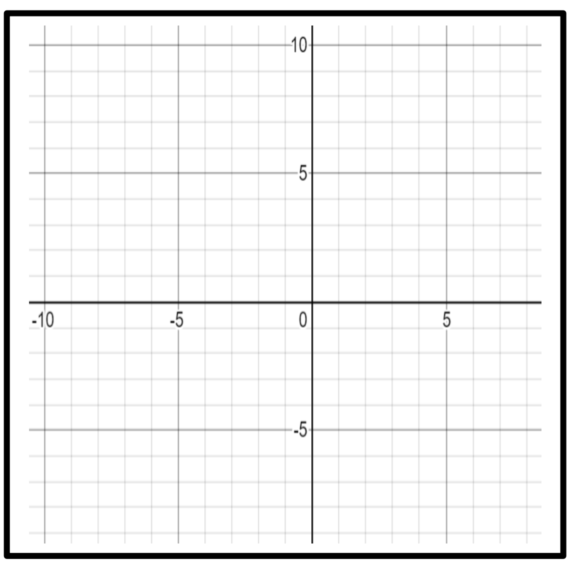
b) $y = 2x^2 + 6x$

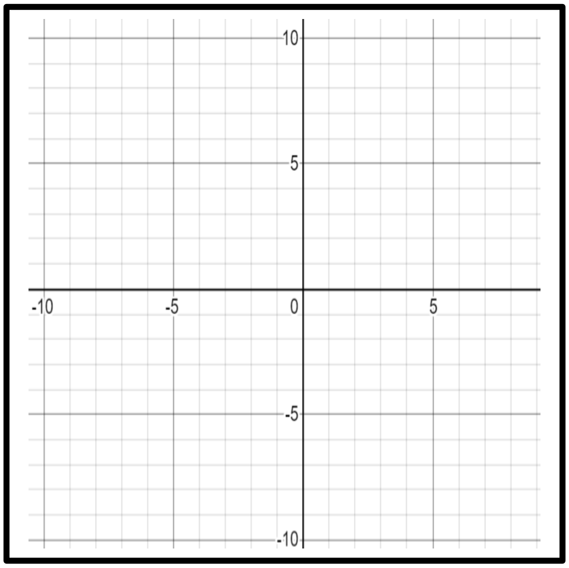


x	$y = 2x^2 + 6x$	(x, y)

y-intercept	
Max or Min? Value?	
Axis of symmetry	
Vertex	
Domain	
Range	

c) $y = -x^2 + 2x + 3$

x	$y = -x^2 + 2x + 3$	(x, y)



y-intercept	
Max or Min? Value?	
Axis of symmetry	
Vertex	
Domain	
Range	

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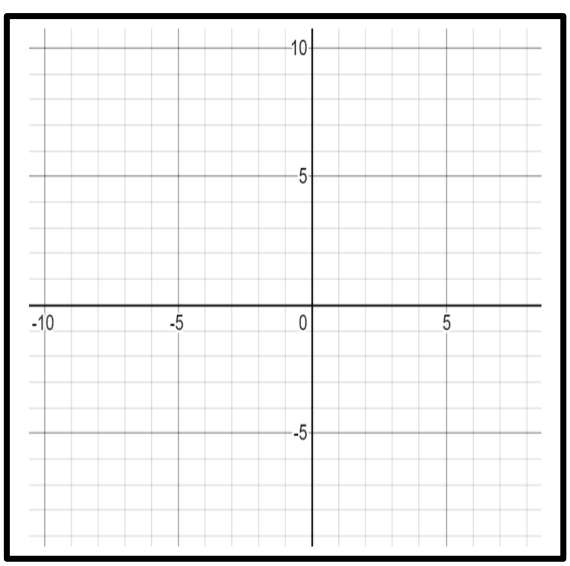
d) $y = x^2 - 6x + 8$

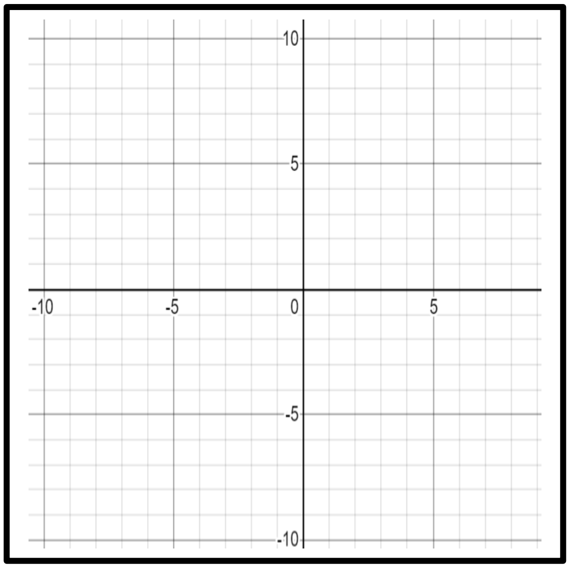
x	$y = x^2 - 6x + 8$	(x, y)



y-intercept	
Max or Min? Value?	
Axis of symmetry	
Vertex	
Domain	
Range	

e) $y = -x^2 + 6x - 9$

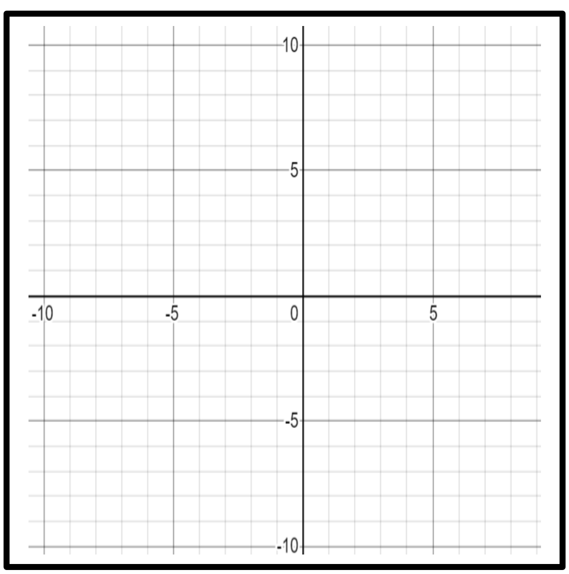
x	$y = -x^2 + 6x - 9$	(x, y)



y-intercept	
Max or Min? Value?	
Axis of symmetry	
Vertex	
Domain	
Range	

f) $y = -\frac{1}{2}x^2 + 4x - 6$

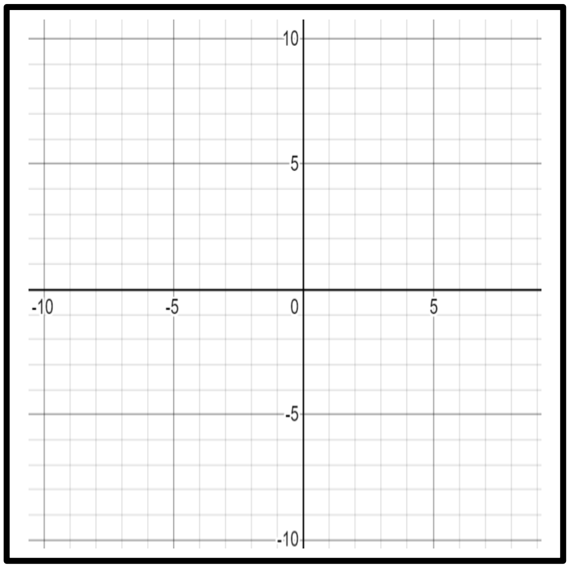
x	$y = -\frac{1}{2}x^2 + 4x - 6$	(x, y)



y-intercept	
Max or Min? Value?	
Axis of symmetry	
Vertex	
Domain	
Range	

g) $y = x^2 + 2x + 3$

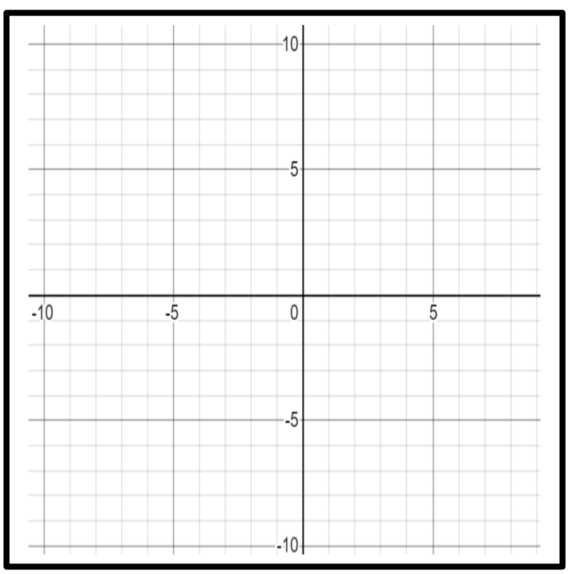
x	$y = x^2 + 2x + 3$	(x, y)



y-intercept	
Max or Min? Value?	
Axis of symmetry	
Vertex	
Domain	
Range	

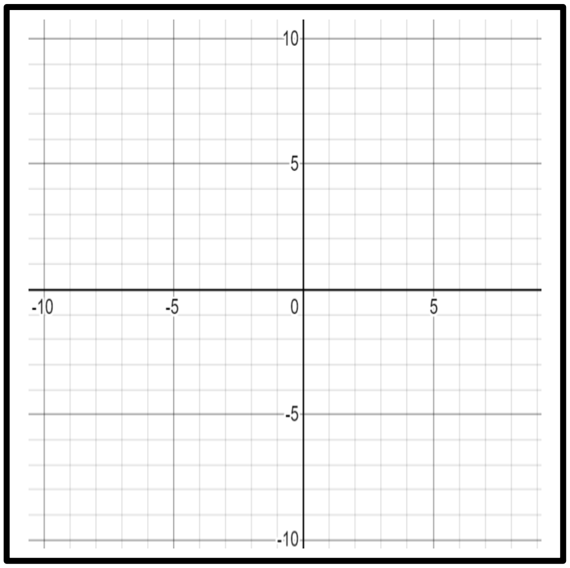
h) $y = -x^2 - 6x - 4$

x	$y = -x^2 - 6x - 4$	(x, y)



y-intercept	
Max or Min? Value?	
Axis of symmetry	
Vertex	
Domain	
Range	

i) $y = 2x^2 - 4x + 1$



x	$y = 2x^2 - 4x + 1$	(x, y)

y-intercept	
Max or Min? Value?	
Axis of symmetry	
Vertex	
Domain	
Range	

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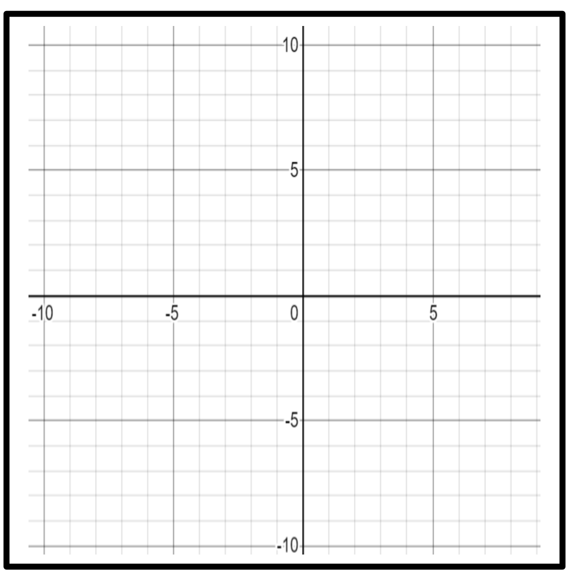
j) $y = -3x^2 - 12x - 8$



x	$y = -3x^2 - 12x - 8$	(x, y)

y-intercept	
Max or Min? Value?	
Axis of symmetry	
Vertex	
Domain	
Range	

VERTEX FORM OF A QUADRATIC FUNCTION

The equation of the basic quadratic function is $y = x^2$ and its graph is a parabola. When changes are made to the equation, the parabola will change.

Example 1:

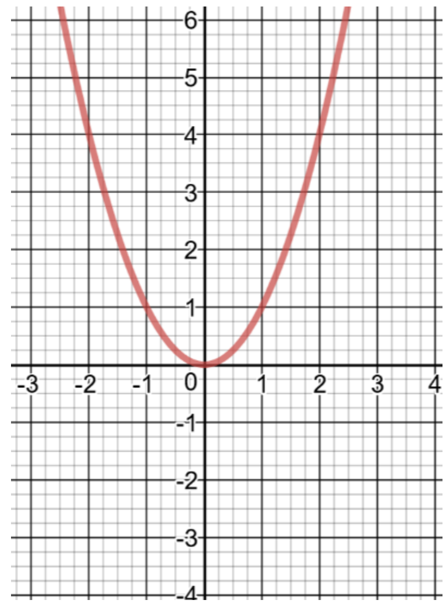
GRAPHING $y = x^2 + k$

The graph of $y = x^2$ is shown.

On the same coordinate plane, graph:

a) $y = x^2 + 2$

b) $y = x^2 - 3$



Example2:

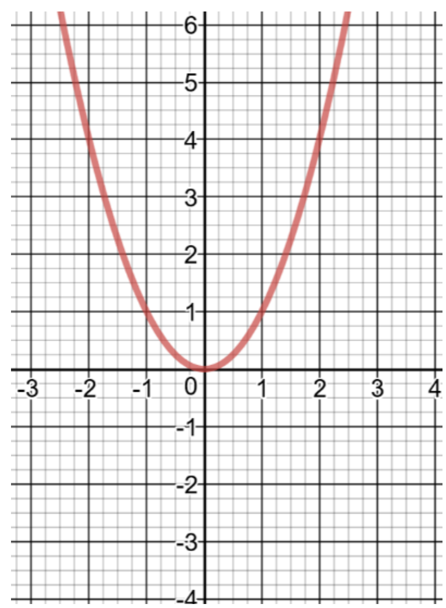
GRAPHING $y = (x - h)^2$

The graph of $y = x^2$ is shown.

On the same coordinate plane, graph:

a) $y = (x + 1)^2$

b) $y = (x - 2)^2$



Example 3:

GRAPHING $y = ax^2$

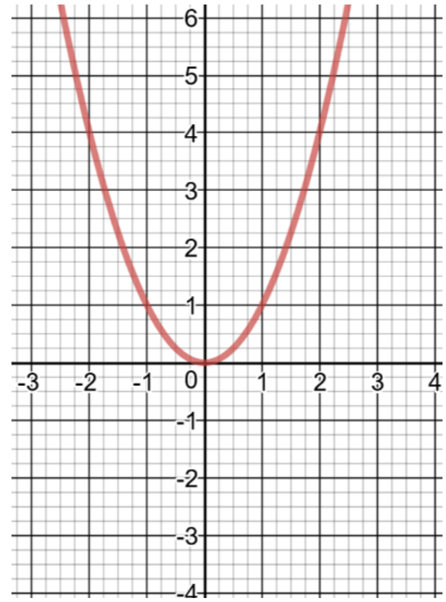
The graph of $y = x^2$ is shown.

On the same coordinate plane, graph:

a) $y = 2x^2$

b) $y = \frac{1}{2}x^2$

c) $y = -x^2$



Combining the forms given in Examples 1,2 and 3 above we get:

$$\boldsymbol{y = a(x - h)^2 + k}$$

This is called the **Vertex Form** or the **Standard Form** of a quadratic function.

To graph a function given in vertex form, use the following steps:

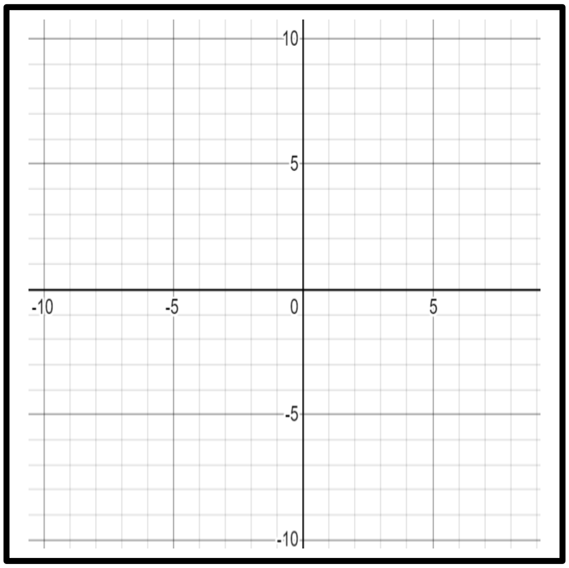
Step 1: The vertex will have coordinates (h, k)

Step 2: Obtain the coordinates of at least two points to the right and two points to the left of the vertex. For this, use two x-values greater than and two x-values less than h.

Step 3: If possible, find the $x - intercepts$ (by letting $y = 0$ and solving for x)

Example

Graph $y = 2(x-4)^2 - 8$ and complete the given table.



x	$y = 2(x-4)^2 - 8$	(x, y)

y-intercept	
Max or Min? Value?	
Axis of symmetry	
Vertex	
Domain	
Range	

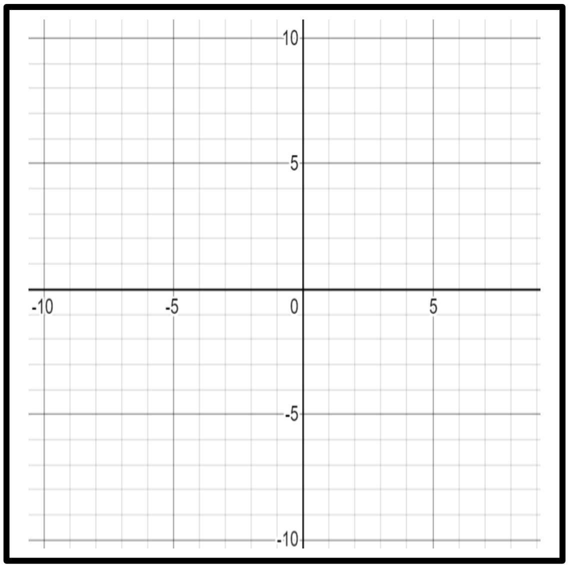
For a function given in Standard Form, the $x - intercepts$ can obtained quite easily. We do this by letting $y = 0$ and then solving for x.

Example

Calculate the $x - intercepts$ for the above example.

Example

Graph $y = -\frac{1}{2}(x+1)^2 + 2$ and complete the given table.

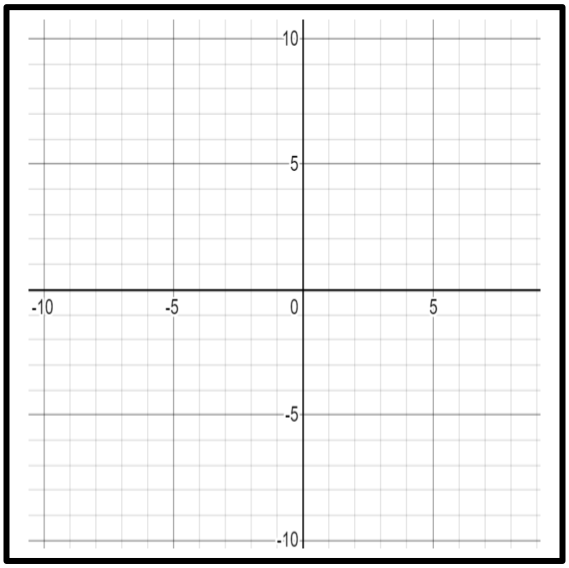


x	$y = -\frac{1}{2}(x+1)^2 + 2$	(x, y)

y −intercept	
Max or Min? Value?	
Axis of symmetry	
x −intercepts	
Domain	
Range	

Example

Graph $y = \frac{1}{3}(x+2)^2$ and complete the given table.



x	$y = \frac{1}{3}(x+2)^2$	(x, y)

y −intercept	
Max or Min? Value?	
Axis of symmetry	
x −intercepts	
Domain	
Range	

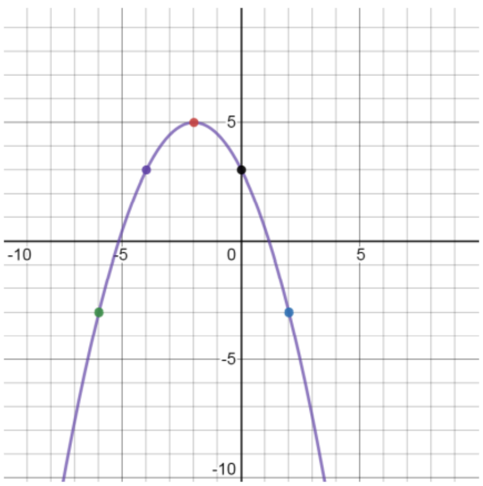
<u>FINDING THE EQUATION OF A PARABOLA FROM A GRAPH</u>

Finding the equation of a parabola from a graph requires two things:

1. The vertex
2. The value that determines the shape and direction of the parabola…the value of a

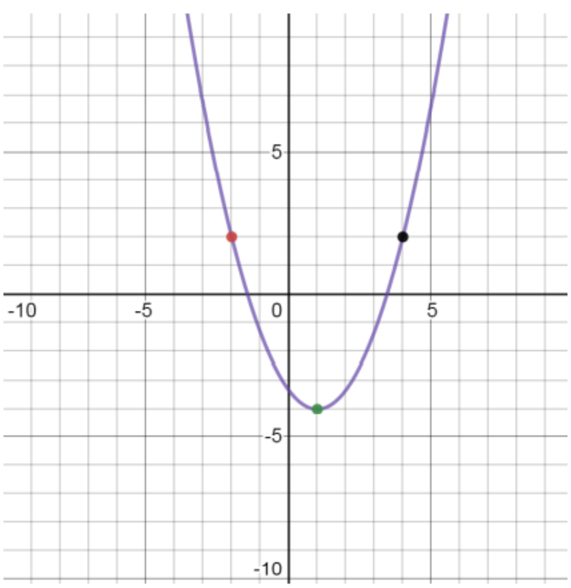
Example

Determine the equation for the given parabola.



Example

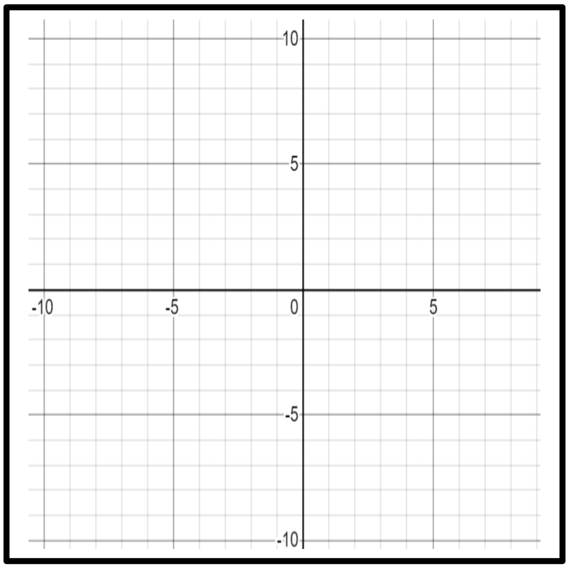
Determine the equation for the given parabola.



PRACTICE PROBLEMS

1. Graph the following. Complete the given table.

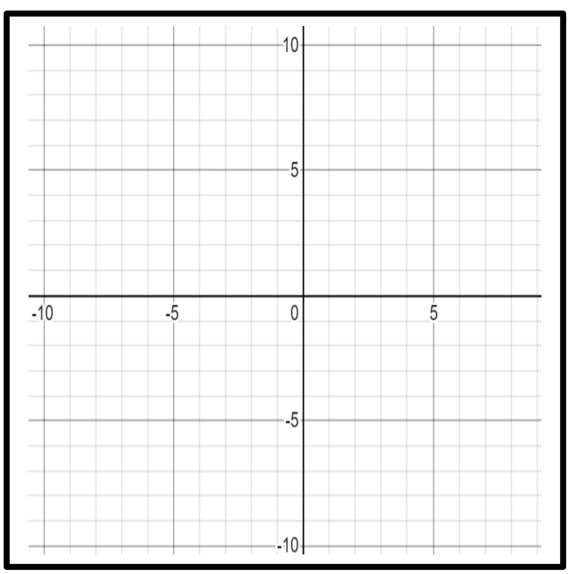
a) $y = (x-2)^2$



x	$y = (x-2)^2$	(x, y)

y −intercept	
Max or Min? Value?	
Axis of symmetry	
x −intercept(s)	
Domain	
Range	

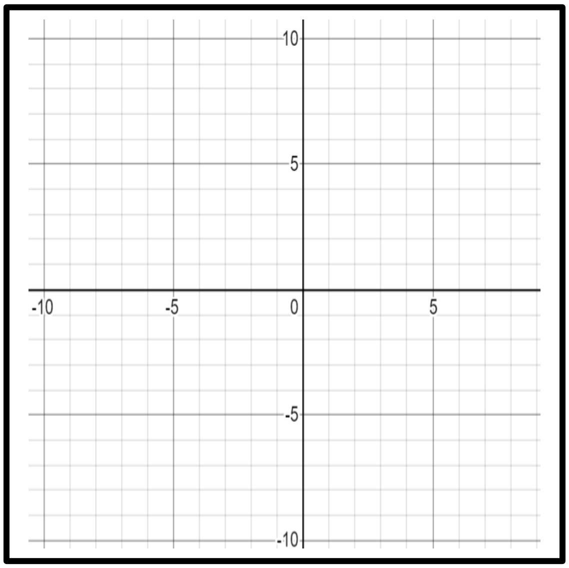
b) $y = -(x+1)^2 + 2$



x	$y = -(x+1)^2 + 2$	(x, y)

y −intercept	
Max or Min? Value?	
Axis of symmetry	
x −intercept(s)	
Domain	
Range	

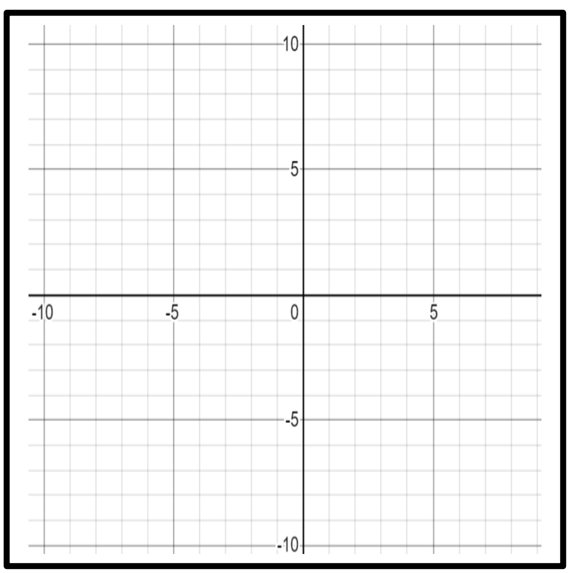
c) $y = \frac{1}{2}(x+2)^2 - 2$



x	$y = \frac{1}{2}(x+2)^2 - 2$	(x, y)

y −intercept	
Max or Min? Value?	
Axis of symmetry	
x −intercept(s)	
Domain	
Range	

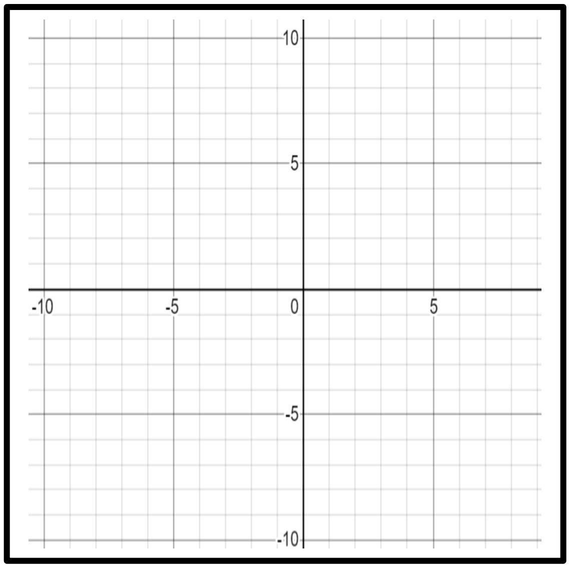
d) $y = -\frac{1}{2}(x-2)^2 + 3$



x	$y = -\frac{1}{2}(x-2)^2 + 3$	(x, y)

y −intercept	
Max or Min? Value?	
Axis of symmetry	
x −intercept(s)	
Domain	
Range	

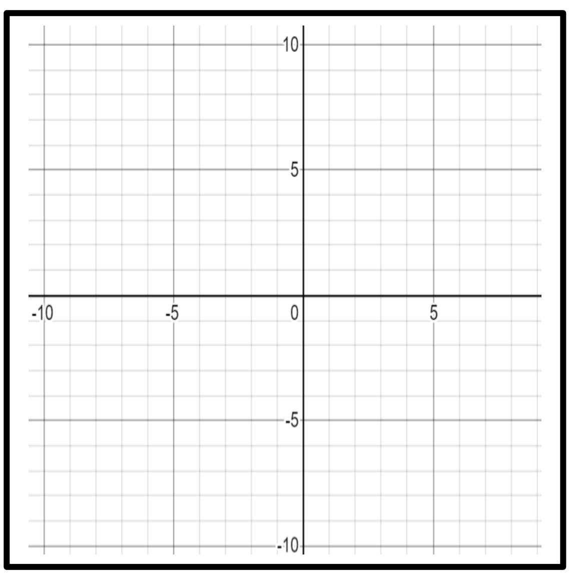
e) $y - 4 = -\frac{2}{3}(x + 2)^2$



x	$y - 4 = -\frac{2}{3}(x + 2)^2$	(x, y)

y −intercept	
Max or Min? Value?	
Axis of symmetry	
x −intercept(s)	
Domain	
Range	

f) $y + 3 = \frac{3}{4}(x - 4)^2$

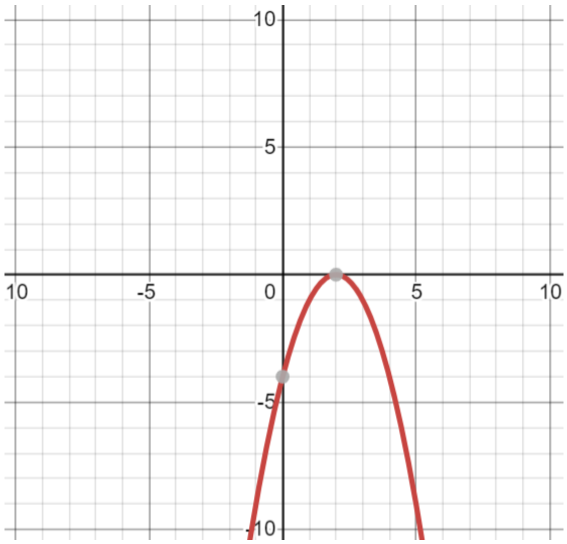


x	$y + 3 = \frac{3}{4}(x - 4)^2$	(x, y)

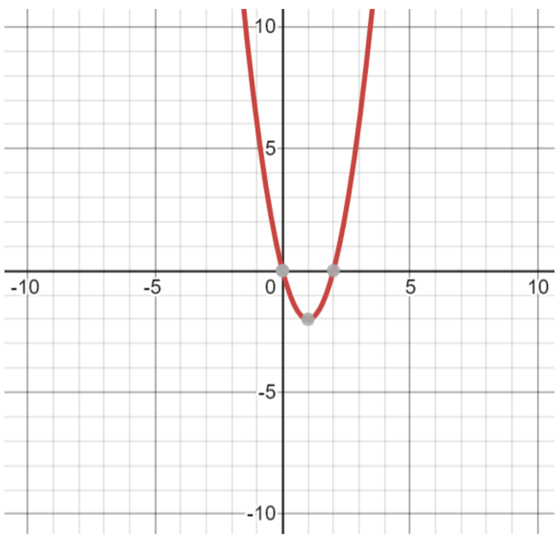
y −intercept	
Max or Min? Value?	
Axis of symmetry	
x −intercept(s)	
Domain	
Range	

2. Determine in vertex form, the equation for the following:

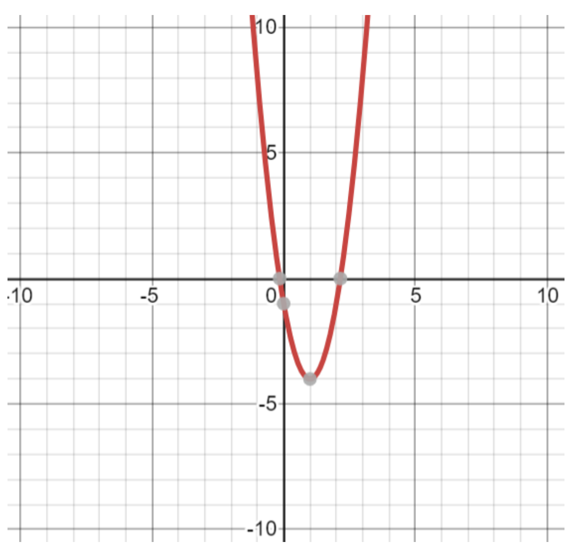
a)



b)



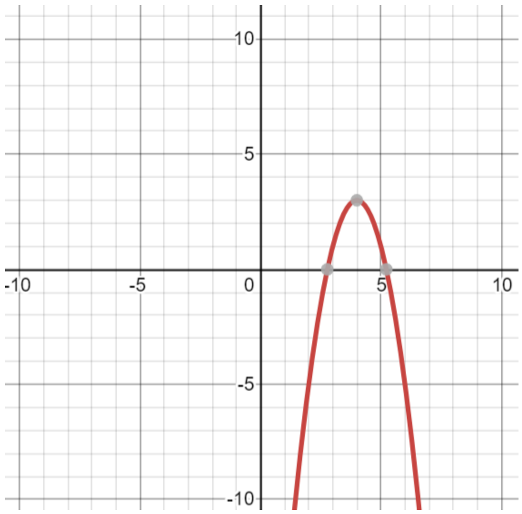
c)



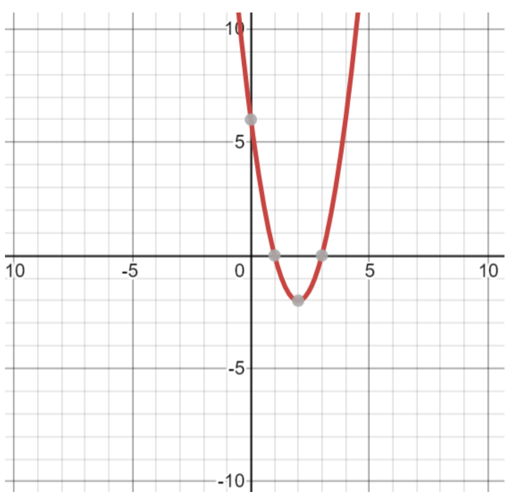
3. Determine the equation for the following in **General Form ($y = ax^2 + bx + c$).**

(First write the equation in Vertex Form and then convert to General Form)

a)



b)



4. Determine the value of *a* given that (7, -6) satisfies the quadratic function

$f(x) = a(x-5)^2 + 6$

5. Determine the equation of a parabola with vertex (1, 7) and point (4, -20).

6. A parabola has the vertex (-7, -2).

a) Write an equation to describe all parabolas with this vertex.

b) A parabola with the given vertex passes through the point (-9, 10). Determine the equation for this parabola.

c) State the domain and range of the function.

QUADRATIC EQUATIONS

Recall the quadratic **functions** can be written in the form $y = ax^2 + bx + c$.

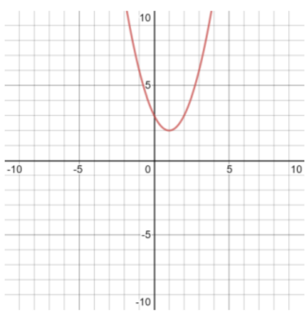
A quadratic **equation** is written in the form $ax^2 + bx + c = 0$,where
a, b and c are real numbers and $a \neq 0$.

In a quadratic equation, we are able to solve for values of x which make the equation equal 0.

- The solutions of the quadratic equation $ax^2 + bx + c = 0$ are called **zeros**, **roots** or **solutions.**
- The **zeros**, **roots** or **solutions** are also called the $\boldsymbol{x - intercepts}$, because y is equal to zero at these points.
- A quadratic function can cross the $x - axis$ either **0, 1 or 2** times. This means the function could have 0, 1 or 2 $x - intercepts$ (or the equation has 0, 1 or 2 roots/ zeros/solutions)

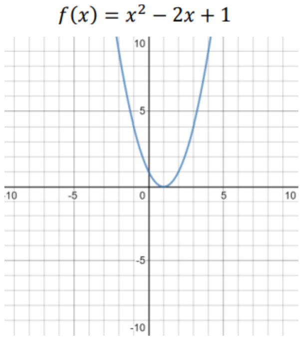
Example

$f(x) = x^2 - 2x + 3$



This has no $x - intercepts$ and therefore no roots (or zeros)

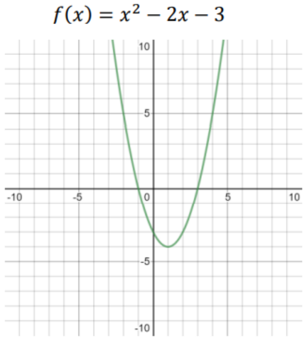
$f(x) = x^2 - 2x + 1$



This has one $x - intercept$ and therefore one root (or zero).

The $x - intercept$ = 1, so the root =1

$f(x) = x^2 - 2x - 3$



This has two $x - intercepts$ and therefore two roots (or zeros).

The $x - intercept = \ -1 \ and \ 3$, so the roots = -1 and 3.

FACTORING AND SOLVING QUADRATIC EQUATIONS

To solve a quadratic equation in the form $ax^2 + bx + c = 0$, we can first factor the equation and then solve the factored equation.

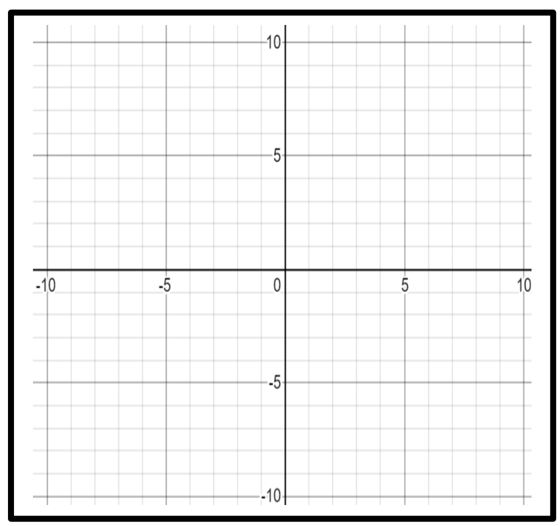
To solve for the roots (or find the $x - intercepts$) we will need to factor equations.

The factoring techniques used are:

- Common Factors: $ax + am = a(x + m)$
- Difference of squares: $a^2 - b^2 = (a + b)(a - b)$
- Trinomials in the form: $x^2 + bx + c = (x + p)(x + q)$
- Trinomials in the form: $ax^2 + bx + c$ (these are a little more difficult to factor)

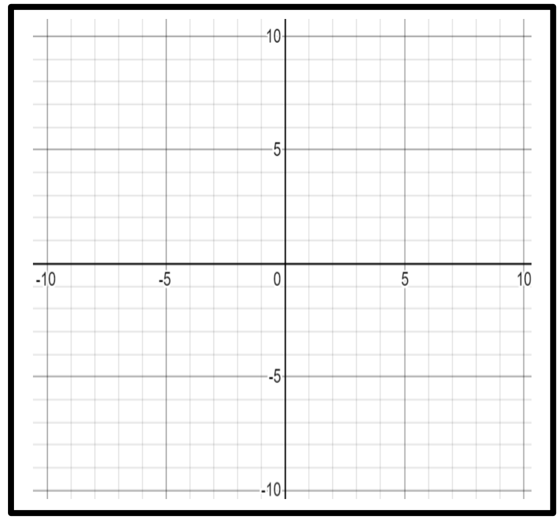
Example

Solve $5x^2 - 10x = 0$ and draw the graph of $y = 5x^2 - 10x$



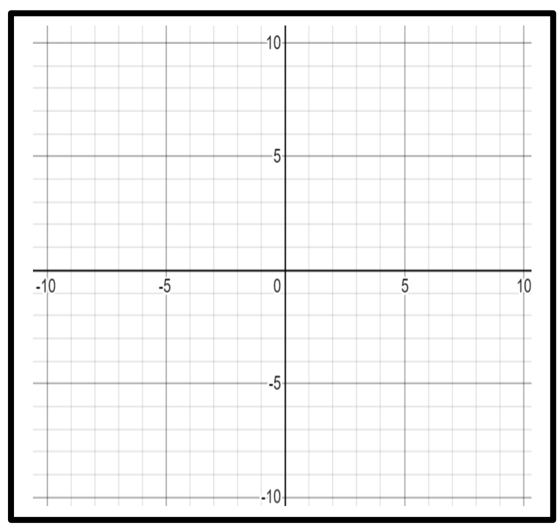
Example

Solve $x^2 - 9 = 0$ and draw the graph of $y = x^2 - 9$



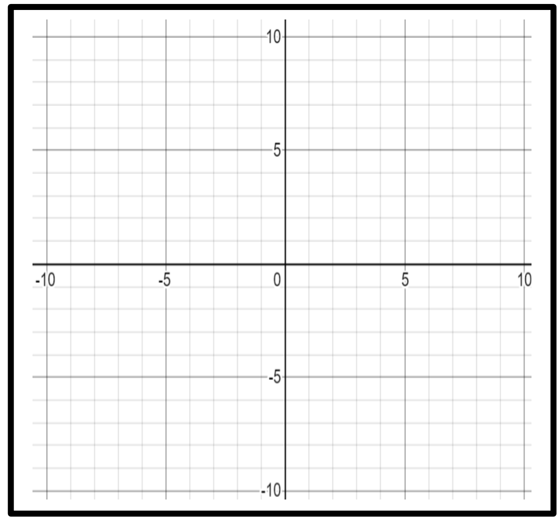
Example

Solve $x^2 - 2x - 3 = 0$ and draw the graph of $y = x^2 - 2x - 3$



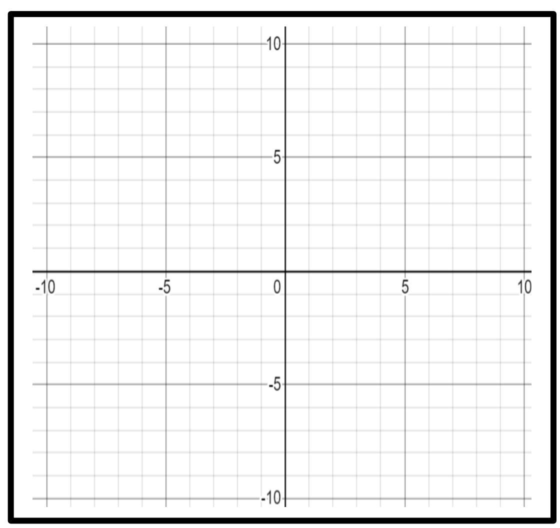
Example

Solve $5x^2 + 35x + 60 = 0$ and draw the graph of $y = 5x^2 + 35x + 60$



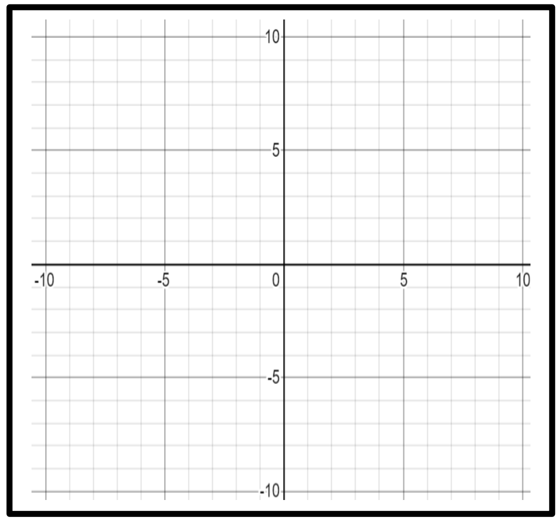
Example

Solve $-x^2 + 3x + 4 = 0$ and draw the graph of $y = -x^2 + 3x + 4$



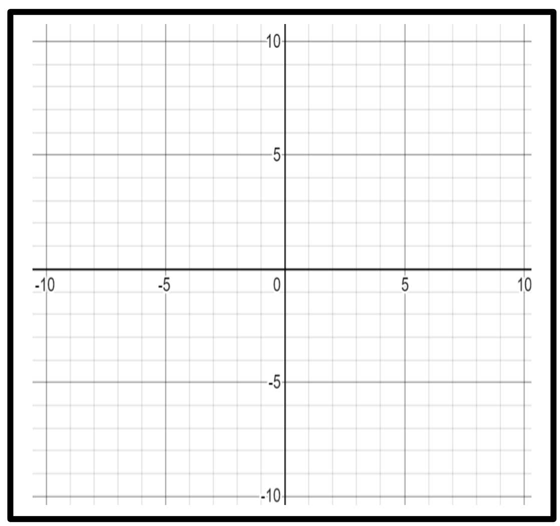
Example

Solve $2x^2 + 3x - 5 = 0$ and draw the graph of $y = 2x^2 + 3x - 5$



Example

Solve $2x^2 - 3x - 5 = 0$ and draw the graph of $y = 2x^2 - 3x - 5$



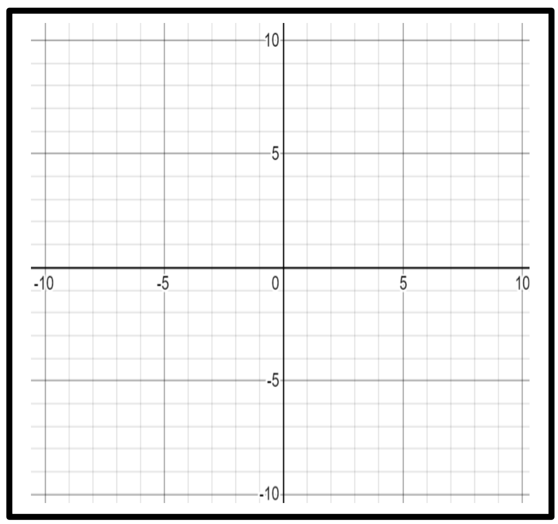
Example

A quadratic function has an equation that can be written in the form $f(x) = a(x - r)(x - s)$. The graph of the function has x-intercepts at (4, 0) and (−2, 0) and passes through the point (–3, –8). Write the equation of the function.

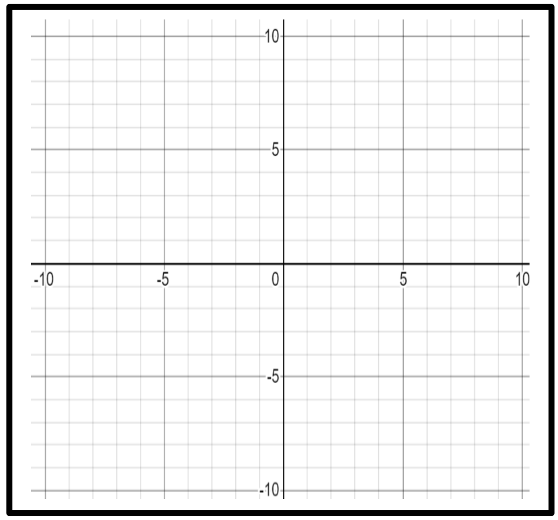
PRACTICE PROBLEMS

1. Solve the following quadratic equations. Then graph the given quadratic function using the roots, vertex and two other coordinates.

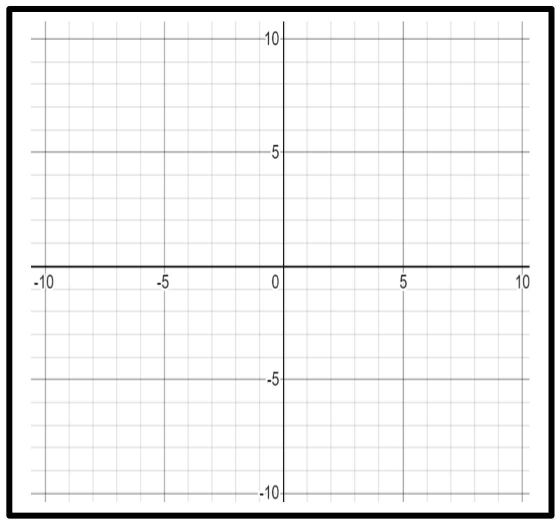
a) Solve $x^2 - 3x = 0$ and draw the graph of $y = x^2 - 3x$



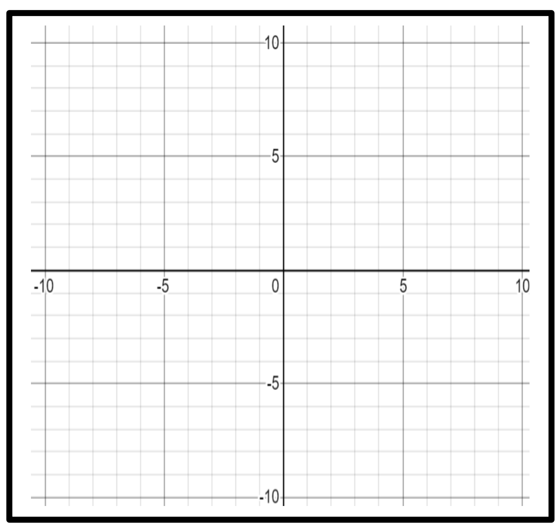
b) Solve $2x^2 - 8 = 0$ and draw the graph of $y = 2x^2 - 8$



c) Solve $2x^2 - x - 6 = 0$ and draw the graph of $y = 2x^2 - x - 6$



d) Solve $4x^2 - 16x = -15$ and draw the graph of $y = 4x^2 - 16x + 15$



e) Solve $3x^2 - 8x = 9 - 2x$ and draw the graph of $y = 3x^2 - 8x + 2x - 9$ (or $y = 3x^2 - 6x - 9$)

f) Solve $8x^2 - 32x = -30$ and draw the graph of $y = 8x^2 - 32x + 30$

3. A quadratic function has an equation that can be written in the form $f(x) = a(x - r)(x - s)$. The graph of the function has x-intercepts at (2, 0) and (−3, 0) and passes through the point (–2, –8). Write the equation of the function.

SOLVING QUADRATIC EQUATIONS USING OTHER METHODS

When a quadratic equation cannot be solved by factoring, we can use the quadratic formula. The quadratic formula can be used to calculate the roots of any quadratic equation.

Quadratic Formula:

The solution to the quadratic equation $ax^2 + bx + c = 0$ is given by:

$$x = \frac{-b \pm\sqrt{b^2-4ac}}{2a}$$

Example: Solve $2x^2 + x - 6 = 0$

Example: Solve $3x^2 + 2x - 4 = 0$

Example: Solve $2x^2 - 3x + 4 = 0$

Example: Solve $2x^2 + 3x - 5 = 0$ (we factored and graphed this equation on page 29)

PRACTICE PROBLEMS

1. Solve using the quadratic formula. Give answers to the nearest hundredth.

a) $x^2 + 5x - 3 = 0$

b) $3x^2 - 6x + 4 = 0$

c) $\frac{3}{2}x^2 - 2 = x$

d) $5x(x+2) = -4$

e) $(x-2)(x-3) = 8$

2. Solve each quadratic equation using either factoring or the quadratic formula (your choice)

a) $4x^2 + 25 = 25x$

b) $25x^2 - 25x + 6 = 0$

c) $(x+3)(x-2) = -4$

d) $2x(x+3) = 10$

APPLICATIONS OF QUADRATIC EQUATIONS

If we can maximize or minimize the quadratic, we can solve many types of problems
Solving for the vertex, and knowing if the vertex is a maximum or a minimum is the key to solving quadratic formula problems

Example:

A punter kicks a football 48 m to another player who catches it. The path of the football is defined by the function:

$$h(x) = -\frac{1}{30}(x - 24)^2 + 19.2$$

where *x* is the horizontal distance measured in metres from the kicker

a) Determine the axis of symmetry of the parabola.

b) What was the highest point of the football's path?

c) What is the range for this function?

Example:

Gordon and Hanna are standing 10 ft apart, playing badminton. They use a video camera to determine that the path of the birdie on one volley is defined by the function

$$h(x) = -0.04(x - 5)^2 + 7$$

where x is the horizontal distance, measured in feet, from Gordon toward Hanna.

a) Determine the axis of symmetry of the parabola.

b) What was the highest point of the birdie's path?

Example:

A soccer ball is kicked from the ground. After 2 seconds, the ball reaches its maximum height of 20 m. It lands on the ground at 4 seconds.

a) Determine the quadratic function that models the height of the kick.

b) Determine any restrictions that must be placed on the domain and range of the function.

c) What was the height of the ball at one second? When was the ball at the same height on the way down?

Example:

A bus company charges $2 per ticket but wants to raise the price. The daily revenue that could be generated is modeled by the function

$$R(x) = -40(x-5)^2 + 25000$$

where x is the number of 10 cent price increases and R is a revenue in dollars. What should the price per ticket be if the bus company wants to collect daily revenue of $21,000?

PRACTICE PROBLEMS

1. A rectangular pen is to be built along the side of a barn. Find the maximum area that can be enclosed with 60 meters of fencing, if the barn is on one side of the enclosure.

2. Bobs Rent- a- Wreck rent 300 cars at $40 per day. For each $1 increase in cost of renting, 5 fewer cars are rented. For what rate should the cars be rented to produce the maximum income? What is the maximum income?

3. Marlene and Candace are both 6 feet tall, and they play on the same college volleyball team. In a game, Candace setup Marlene with an outside highball for an attack hit. Using a video of the game, their coach determined that the height of the ball above the court, in feet, on its path from Candace to Marlene could be defined by the function

$$h(x) = -0.03(x-9)^2 + 8$$

Where x is the horizontal distance, measured in feet, from one edge of the court.

a) Determine the axis of symmetry of the parabola.

b) Marlene hit the ball at its highest point. How high above the court was the ball when she hit it?

c) How high was the ball when Candace set it, if she was 2 feet from the edge of the court?

d) State the range for the balls path between Candace and Marlene.

4. Susan and a friend are throwing a paper airplane to each other. They stand 5 m apart from each other and catch the airplane at a height of 1 m above the ground.
Susan throws the airplane on a parabolic flight path that achieves a minimum height of 0.5 meters halfway to her friend.

a) Determine a quadratic function that models the flight path for the height of the airplane.

b) Determine the height of the plane when it is a horizontal distance of 1 m from Susan's friend.

c) State the domain and range of the function

5. Duncan dives with a junior swim club. In a dive of a 7.5 m platform, he reaches a maximum height of 7.94 meters after 0.30 seconds. How long does it take him to reach the water?