## 4.3 Rational Exponents

### KEY IDEAS

- A power with a negative exponent can be written as a power with a positive exponent.

  – $a^{-n} = \frac{1}{a^n}, a \neq 0$ $\quad 9^{-1.3} = \frac{1}{9^{1.3}}$ $\quad$ – $\frac{1}{a^{-n}} = a^n, a \neq 0$ $\quad \frac{1}{2^{-3.2}} = 2^{3.2}$

- You can apply the above principle to the exponent laws.

| Exponent Law | Example |
|---|---|
| Note that $a$ and $b$ are rational or variable bases and $m$ and $n$ are integral exponents. | |
| Product of Powers<br>$(a^m)(a^n) = a^{m+n}$ | $\left(x^{\frac{3}{5}}\right)\left(x^{\frac{6}{5}}\right) = x^{\frac{3}{5}+\frac{6}{5}}$<br>$= x^{\frac{9}{5}}$ |
| Quotient of Powers<br>$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$ | $\frac{4s^{2.5}}{12s^{0.5}} = \frac{1}{3}s^{(2.5-0.5)}$<br>$= \frac{1}{3}s^2$ or $\frac{s^2}{3}$ |
| Power of a Power<br>$(a^m)^n = a^{mn}$ | $(t^{3.3})^{\frac{1}{3}} = t^{(3.3)\left(\frac{1}{3}\right)}$<br>$= t^{1.1}$ |
| Power of a Product<br>$(ab)^m = a^m b^m$ | $\left(8x^{\frac{1}{2}}\right)^{\frac{2}{3}} = (2^3)^{\frac{2}{3}}\left(x^{\frac{1}{2}}\right)^{\frac{2}{3}}$<br>$= 4x^{\frac{2}{6}}$ or $4x^{\frac{1}{3}}$ |
| Power of a Quotient<br>$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \neq 0$ | $\left(\frac{x^3}{y^6}\right)^{\frac{1}{3}} = \frac{(x^3)^{\frac{1}{3}}}{(y^6)^{\frac{1}{3}}}$<br>$= \frac{x}{y^2}$ |
| Zero Exponent<br>$a^0 = 1, a \neq 0$ | $(-2y^2)^0 = 1$<br>$-(2y^2)^0 = -1$ |

- A power with a rational exponent can be written with the exponent in decimal or fractional form. $\quad x^{\frac{3}{5}} = x^{0.6}$

## Example

Write each product or quotient as a power with a single positive exponent. Then, evaluate where possible.

**a)** $\left(7^{\frac{1}{2}}\right)(7^3)$ **b)** $\frac{9^{1.25}}{9^{\frac{3}{4}}}$ **c)** $(16x^6)^{\frac{1}{4}}$ **d)** $\left(\frac{3^{0.25}}{3^{\frac{3}{4}}}\right)^3$ **e)** $\left(\frac{27}{8}\right)^{-0.4}$

## Solution

**a)** Since the bases are the same, you can add the exponents. Remember to determine the lowest common denominator when adding fractions.

$$\left(7^{\frac{1}{2}}\right)(7^3) = \left(7^{\frac{1}{2}}\right)\left(7^{\frac{6}{2}}\right)$$
$$= 7^{\left(\frac{1}{2}+\frac{6}{2}\right)}$$
$$= 7^{\frac{7}{2}}$$

**b)** Convert the rational exponents so both are fractions or decimal numbers. Then, since the bases are the same, you can subtract the exponents.

**Method 1: Convert to Fractions**

$$\frac{9^{1.25}}{9^{\frac{3}{4}}} = \frac{9^{\frac{5}{4}}}{9^{\frac{3}{4}}}$$
$$= 9^{\left(\frac{5}{4} - \frac{3}{4}\right)}$$
$$= 9^{\frac{2}{4}}$$
$$= 9^{\frac{1}{2}}$$
$$= 3$$

**Method 2: Convert to Decimals**

$$\frac{9^{1.25}}{9^{\frac{3}{4}}} = \frac{9^{1.25}}{9^{0.75}}$$
$$= 9^{(1.25 - 0.75)}$$
$$= 9^{0.5}$$
$$= 3$$

**c)** Raise each term to the exponent. Then, multiply the exponents.

$$(16x^6)^{\frac{1}{4}} = (16)^{\frac{1}{4}}\, x^{(6)\left(\frac{1}{4}\right)}$$
$$= 2x^{\frac{6}{4}}$$
$$= 2x^{\frac{3}{2}}$$

**d)** **Method 1: Subtract the Exponents**

Convert the rational exponents to fractions or decimal numbers. Since the bases are the same, you can subtract the exponents. Raise the result to the exponent 3. Then, multiply.

Convert to fractions:

$$\left(\frac{3^{0.25}}{3^{\frac{3}{4}}}\right)^3 = \left(\frac{3^{\frac{1}{4}}}{3^{\frac{3}{4}}}\right)^3$$
$$= \left[3^{\left(\frac{1}{4} - \frac{3}{4}\right)}\right]^3$$
$$= \left(3^{\frac{-2}{4}}\right)^3$$
$$= \left(3^{\frac{-1}{2}}\right)^3$$
$$= \left(3^{\frac{-3}{2}}\right)$$
$$= \frac{1}{3^{\frac{3}{2}}}$$

**Method 2: Apply Power of a Power**

Raise each power to the exponent 3. Next, convert the rational exponents to fractions or decimal numbers. Then, subtract the exponents of the resulting powers.

$$\left(\frac{3^{0.25}}{3^{\frac{3}{4}}}\right)^3 = \left(\frac{3^{\frac{3}{4}}}{3^{\frac{9}{4}}}\right)$$
$$= 3^{\frac{3}{4} - \frac{9}{4}}$$
$$= 3^{\frac{-6}{4}}$$
$$= \frac{1}{3^{\frac{3}{2}}}$$

**e)** Convert the bases to a single exponent. Then, raise the result to the exponent −0.4.

$$\left(\frac{27}{8}\right)^{-0.4} = \left[\frac{(3^3)}{(2^3)}\right]^{-0.4}$$
$$= \left[\left(\frac{3}{2}\right)^3\right]^{-0.4}$$
$$= \left(\frac{3}{2}\right)^{-1.2}$$
$$= \left(\frac{2}{3}\right)^{1.2}$$

## A Practise

1. Use the exponent laws to simplify each expression. Where possible, compute numerical values.

**a)** $(a^6)(a^{\frac{3}{2}})$ **b)** $(y^{\frac{1}{3}})(y^{\frac{1}{2}})$

**c)** $(x^{0.4})(x^{\frac{1}{2}})$ **d)** $(a^{0.2})^3$

**e)** $(x^{\frac{2}{3}})^{-6}$ **f)** $(81^{\frac{1}{4}})^2$

**g)** $\left(\frac{-64x^{\frac{3}{4}}}{27x^{\frac{1}{2}}}\right)^{\frac{1}{3}}$ **h)** $(-5a^{\frac{1}{2}})(2a^{\frac{3}{5}})$

**i)** $(256a^6)^{0.25}$

2. Use the exponent laws to simplify each expression. Leave your answers with positive exponents.

**a)** $(a^{-2})(a^{\frac{3}{4}})$ **b)** $(16^{-0.25})^2$

**c)** $\frac{(y^{\frac{2}{3}})^{-2}}{(y^{\frac{1}{2}})^{-4}}$ **d)** $(a^{\frac{3}{4}})^{-0.5}(a^2)^{-0.25}$

**e)** $\left[\frac{(a^2b)}{(ab)^3}\right]^{-1.5}$ **f)** $\left(\frac{25x^{-2}}{16x^{\frac{-1}{2}}}\right)^{-1.5}$

**g)** $(4x^3)^{\frac{-1}{2}}(27y^2)^{\frac{1}{3}}$ **h)** $\left(\frac{81x^{\frac{2}{3}}}{625y^{\frac{3}{5}}}\right)^{0.25}$

☆3. Use the exponent laws to help identify a value for $q$ that satisfies each equation.

**a)** $(x^{\frac{2}{3}})^q = x^{\frac{4}{3}}$ **b)** $(x^{\frac{-2}{3}})(x^q) = x^{\frac{-1}{6}}$

**c)** $\frac{y^{\frac{2}{3}}}{y^q} = y^{\frac{11}{12}}$ **d)** $(27x^2)^{\frac{1}{3}}(qx^2)^{\frac{-1}{2}} = \frac{3}{2x^{\frac{1}{3}}}$

**e)** $(5^q)(-3^{-q}) = \frac{-125}{27}$

4. Evaluate without using a calculator. Leave the answers as rational numbers.

**a)** $16^{\frac{3}{4}}$ **b)** $-243^{\frac{2}{5}}$

**c)** $8^{\frac{-5}{3}}$ **d)** $\left(\frac{49}{9}\right)^{\frac{3}{2}}$

**e)** $\left(\frac{125x^2}{8y^3}\right)^{\frac{2}{3}}$ **f)** $\frac{5^{-2}}{25^{\frac{-3}{2}}}$

5. Evaluate using a calculator. Express the answers to four decimal places, if necessary.

**a)** $(9^{-0.5})^3$ **b)** $(64^{\frac{1}{2}})^3$

**c)** $(3^{1.2})(3^{2.4})$ **d)** $\frac{16^{\frac{2}{3}}}{16^{-0.2}}$

**e)** $\left(\frac{3^{\frac{-1}{2}}}{81^{0.25}}\right)^2$ **f)** $\left(\frac{8}{7^{\frac{1}{2}}}\right)^{\frac{5}{3}}$

6. Mid Lake, in Manitoba, is stocked with rainbow trout annually. The population grows at a rate of 11.5% per month. The number of trout stocked is given by the expression $623(1.115)^n$, where $n$ is the number of months since the start of the trout season. Determine the number of trout after

**a)** 3 months **b)** $7\frac{1}{2}$ months

**c)** $3\frac{1}{2}$ months **d)** $6\frac{1}{2}$ months

## B Apply

☆7. For each solution, identify the step where an error was made. What is the correct answer? Compare your corrections with those of a classmate.

**a)** $\frac{a^{\frac{2}{3}}}{a^{\frac{1}{4}}} = a^{\frac{2}{3}-\frac{1}{4}}$
$= a^{-1}$
$= \frac{1}{a}$

**b)** $(16y^{-6})^{-0.5} = (16)^{-0.5}(y^{-6})^{-0.5}$
$= 8y^3$

8. Karen has saved $1500 for college. She deposits this amount into a 3-year term deposit that earns 3.25% interest per year. The formula for calculating the value of her investment is $A = P(1 + i)^n$, where $A$ is the amount of money at the end of the term, $i$ is the interest rate as a decimal number, and $n$ is the number of years the money is invested. How much will her investment be worth at the end of

**a)** 3 years?

**b)** $2\frac{1}{2}$ years?

9. A species of bacteria increases in number by 50% every 25 min. The growth of the bacteria can be modelled using the equation $N = 1000(1.5)^{\frac{t}{25}}$, where $N$ is the number of bacteria after $t$ min.

   **a)** What does the value 1.5 in the formula represent? the value 1000?

   **b)** How many bacteria are present after 1 h?

   **c)** How many bacteria were present 30 min ago?

10. In June 2009, there were approximately 33.985 million people living in Canada. Assume that Canada's natural growth rate is 0.3% per year and that this growth rate remains constant. The natural growth rate represents the number of births and number of deaths in a population and does not take immigration and emigration into account. Canada's growth rate can be modelled using the formula $P = 33.985(1.003)^n$, where $P$ is the population in millions and $n$ is the number of years since 2009.

    **a)** What is the projected population in 15.5 years? Express the answer to three decimal places.

    **b)** What was Canada's population in March 2001?

☆**11.** Bismuth-214 has a half-life of approximately 20 min. The amount of bismuth-214 remaining in a sample that began at 28 g can be represented by the formula $A = 28(0.5)^{\frac{t}{20}}$, where $A$ is the amount remaining after $t$ min. Determine the amount of bismuth-214 remaining after each of the following periods of time. Express each answer to the nearest hundredth of a gram.

    **a)** after 45 min

    **b)** after 2 h

    **c)** after $3\frac{1}{4}$ h

12. In the formula $C = 50(0.5)^{\frac{t}{3}}$, $C$ is the remaining concentration of a particular medication in the bloodstream, in milligrams, and $t$ is the time, in hours.

    **a)** Determine the missing values in the table.

| **Time ($t$)** | 0 | 3 | 6 | 9 | 12 |
|---|---|---|---|---|---|
| **Concentration ($C$)** | | | | | |

    **b)** Graph the data in part a). Let $t$ represent the $x$-axis and $C$ represent the $y$-axis.

    **c)** If $C = 0.195$ mg, what is the value of $t$?

    **d)** If $t = 42$ h, what is the value of $C$?

## C Extend

13. Phosphorus-32 has a half-life of 14 days. If 2.56 g of a sample of phosphorus-32 remain after 70 days, what was the original mass of the sample? Use the formula $A_f = A_i(0.5)^{\frac{t}{14}}$, where $A_f$ is the final amount, $A_i$ is the initial amount, and $t$ is the time in days.

14. Michelle invested money in a mutual fund. By the end of 3 years, she had lost 8% of the original value of her investment. Her account balance was $2672.57.

    **a)** How much did Michelle originally invest? Use the formula $A_f = A_i(1 - r)^{\frac{t}{12}}$, where $A_f$ is the final value of her investment, $A_i$ is the original amount invested, $r$ is the percent of the value lost, and $t$ is the time, in months.

    **b)** How much money did she lose in her investment?

## D Create Connections

☆**15.** Solve for $x$ using the exponent laws.

$4^{\frac{1}{2}} + 4^{\frac{1}{2}} + 4^{\frac{1}{2}} + 4^{\frac{1}{2}} = 4^x$