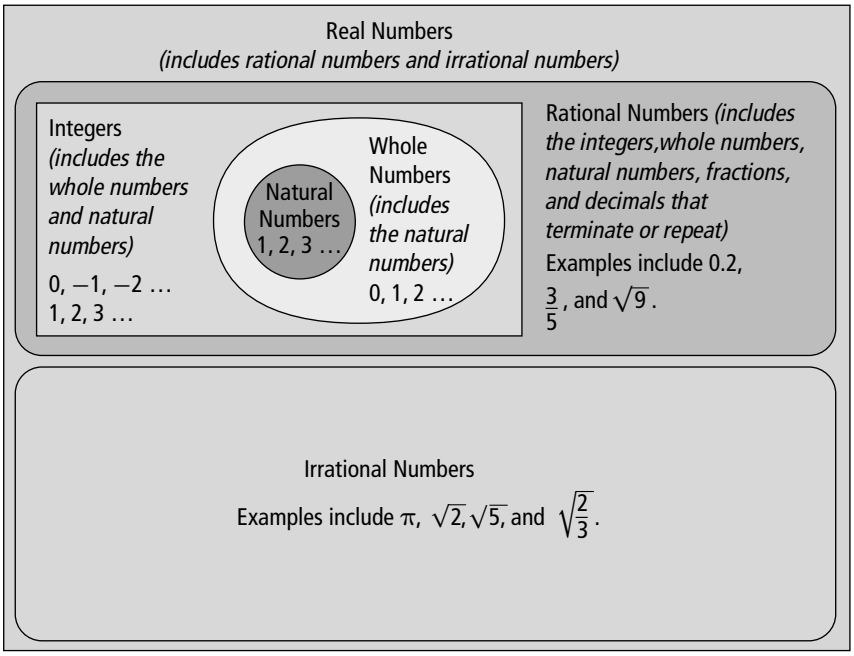
## 4.4 Irrational Numbers

### KEY IDEAS

- Rational numbers and irrational numbers form the set of real numbers.




- Radicals can be expressed as powers with fractional exponents.
  $\sqrt[n]{x^m} = x^{\frac{m}{n}}$
  The index of the radical has the same value as the denominator of the fractional exponent.
  $\sqrt[3]{10} = 10^{\frac{1}{3}} \qquad \sqrt[5]{7^3} = 7^{\frac{3}{5}}$

- Radicals can be entire radicals such as $\sqrt{72}$, $\sqrt[5]{96}$, and $\sqrt[3]{\frac{54}{8}}$. They can also be mixed radicals such as $6\sqrt{2}$, $2\sqrt[5]{3}$, and $\frac{3\sqrt{2}}{2}$. You can convert between entire radicals and mixed radicals.

- You can order radicals that are irrational numbers using different methods:
  - Use a calculator to produce approximate values.
  - Express each irrational number as an entire radical.

## Example

Convert each of the following as requested.
Express each power as an equivalent radical.

**a)** $32^{\frac{1}{2}}$ **b)** $16^{\frac{2}{3}}$ **c)** $(8x^3)^{\frac{1}{4}}$

Express each radical as a power with a rational exponent.

**d)** $\sqrt{6^3}$ **e)** $\sqrt[3]{5^2}$ **f)** $\sqrt[t]{8^3}$

Express each mixed radical as an entire radical.

**g)** $2.5\sqrt{4}$ **h)** $2\sqrt[3]{4}$ **i)** $-2\sqrt[5]{3}$

Express each entire radical as a mixed radical.

**j)** $\sqrt{112}$ **k)** $\sqrt[4]{96}$ **l)** $\sqrt{252}$

## Solution

Write each power as a radical. Use the denominator of the exponent as the index.

**a)** $32^{\frac{1}{2}} = \sqrt{32}$ **b)** $16^{\frac{2}{3}} = \left(\sqrt[3]{16}\right)^2$ **c)** $(8x^3)^{\frac{1}{4}} = \sqrt[4]{8x^3}$

Write each radical as a power. Use the index as the denominator of the exponent.

**d)** $\sqrt{6^3} = 6^{\frac{3}{2}}$ **e)** $\sqrt[3]{5^2} = 5^{\frac{2}{3}}$ **f)** $\sqrt[t]{8^3} = 8^{\frac{3}{t}}$

The index is 3. Convert the whole number 2 to a radical using the fraction $\frac{3}{3}$.
$2^{\frac{3}{3}} = \sqrt[3]{(2^3)}$

Write each mixed radical as an entire radical.

**g)**
$$\begin{aligned} 2.5\sqrt{4} &= \sqrt{(2.5^2)}\sqrt{4} \\ &= \sqrt{(2.5^2)(4)} \\ &= \sqrt{(6.25)(4)} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

**h)**
$$\begin{aligned} 2\sqrt[3]{4} &= \left(2^{\frac{3}{3}}\right)\left(\sqrt[3]{4}\right) \\ &= \sqrt[3]{(2^3)(4)} \\ &= \sqrt[3]{(8)(4)} \\ &= \sqrt[3]{32} \end{aligned}$$

**i)**
$$\begin{aligned} -2\sqrt[5]{3} &= (-2)^{\frac{5}{5}}\left(\sqrt[5]{3}\right) \\ &= \sqrt[5]{(-2^5)(3)} \\ &= \sqrt[5]{(-32)(3)} \\ &= \sqrt[5]{-96} \end{aligned}$$

Express each entire radical as a mixed radical.

**j)**
$$\begin{aligned} \sqrt{112} &= \sqrt{(16)(7)} \\ &= \sqrt{16}\sqrt{7} \\ &= 4\sqrt{7} \end{aligned}$$

**k)**
$$\begin{aligned} \sqrt[4]{96} &= \sqrt[4]{(2)(2)(2)(2)(6)} \\ &= \sqrt[4]{(2^4)(6)} \\ &= 2\sqrt[4]{6} \end{aligned}$$

**l)** $\sqrt{252} = \sqrt{(36)(7)}$ *or* $\sqrt{252} = \sqrt{(4)(63)}$

$$\begin{aligned} &= \sqrt{36}\sqrt{7} && = \sqrt{4}\left(\sqrt{(9)(7)}\right) \\ &= 6\sqrt{7} && = (2)(\sqrt{9})(\sqrt{7}) \\ & && = (2)(3)(\sqrt{7}) \\ & && = 6\sqrt{7} \end{aligned}$$

## A Practise

**1.** Express each power as an equivalent radical.

**a)** $5^{\frac{2}{3}}$ **b)** $8^{0.75}$

**c)** $6^{\frac{3}{5}}$ **d)** $81^{0.5}$

☆**e)** $\frac{1}{9^{\frac{5}{3}}}$ **f)** $(x^3)^{\frac{1}{4}}$

**g)** $\left(a^{\frac{1}{3}}\right)^2$ **h)** $\left[\frac{\left(x^{\frac{1}{3}}\right)}{\left(y^{\frac{1}{3}}\right)}\right]^2$

**2.** Express each radical as a power.

**a)** $\sqrt[4]{3^3}$ **b)** $\sqrt[3]{(5t)^4}$

**c)** $\sqrt[3]{x^2}$ **d)** $\sqrt[5]{\frac{a^2}{b^3}}$

**e)** $\sqrt[3]{y^{\frac{5}{2}}}$ **f)** $\sqrt[a]{2^3}$

**3.** Evaluate each expression. State the result to four decimal places, if necessary.

**a)** $\sqrt{0.25}$ **b)** $(64)^{\frac{1}{3}}$

**c)** $3\sqrt{12}$ **d)** $\sqrt{\left(\frac{5}{4}\right)^2}$

**e)** $4(1.2)^{\frac{3}{4}}$ **f)** $\frac{\sqrt[3]{16}}{\sqrt{12}}$

**4.** Express each mixed radical as an equivalent entire radical.

☆**a)** $4\sqrt{5}$ **b)** $3\sqrt{4}$

**c)** $5\sqrt{13}$ **d)** $6.2\sqrt{10}$

**e)** $3.3\sqrt{16}$ **f)** $\frac{1}{5}\sqrt{10}$

**5.** Express each mixed radical as an equivalent entire radical.

**a)** $3\sqrt[3]{5}$ **b)** $7\sqrt[3]{3}$

**c)** $5\sqrt[3]{6}$ **d)** $2\sqrt[4]{7}$

**e)** $\frac{1}{2}\sqrt[3]{5}$ **f)** $1.5\sqrt[4]{10}$

**6.** Express each entire radical as an equivalent mixed radical.

**a)** $\sqrt{32}$ **b)** $\sqrt{44}$

**c)** $\sqrt{90}$ **d)** $\sqrt{80}$

**e)** $\sqrt{360}$ **f)** $\sqrt{475}$

**7.** Express each entire radical as an equivalent mixed radical.

**a)** $\sqrt[3]{48}$ **b)** $\sqrt[3]{120}$

**c)** $\sqrt[3]{324}$ **d)** $\sqrt[4]{48}$

**e)** $\sqrt[4]{405}$ **f)** $\sqrt[4]{208}$

**8.** Order each set of numbers from greatest to least. Then, identify the irrational numbers.

**a)** $0.5\sqrt{2}$ $0.\overline{7}$ $\frac{3}{4}$ $\sqrt{0.49}$

**b)** $\frac{2}{3}$ $\sqrt[3]{0.343}$ $\sqrt{0.38}$ $0.62$

**9.** Plot each set of numbers on a number line. Which of the numbers in each set is irrational?

**a)** $\sqrt[3]{435}$ $8.\overline{5}$ $4\sqrt{5}$ $\sqrt{64}$

**b)** $\frac{2\sqrt{85}}{3}$ $\sqrt[3]{216}$ $6\frac{9}{11}$ $3\sqrt{7}$

## B Apply

**10.** Determine the diameter of a sphere that has a surface area of 320 cm$^2$. Use the formula $SA = 4\pi r^2$. Express the answer to three decimal places.

**11.** The volume of a cylinder is 312 cm$^3$ and its height is 6 cm. Determine the diameter of the cylinder. Use the formula $V = \pi r^2 h$. Express the answer to the nearest hundredth of a centimetre.

☆**12.** There are approximately 1.3 billion km$^3$ of water on Earth. What would be the length of the edge of a cube that contained Earth's estimated total volume of water? Express the answer to the nearest kilometre.

**13.** In the formula $r = \sqrt[3]{\frac{3V}{4\pi}}$, $r$ represents the radius of a sphere, in centimetres, and $V$ is the volume of the sphere, in cubic centimetres. What is the length of the radius of a sphere with each of the following volumes? Express the answers to two decimal places.

**a)** 132 cm$^3$ **b)** 1896 cm$^3$

**14.** A pendulum has a length of 6 ft. The formula $T = \sqrt{\frac{4\pi^2 l}{32 \text{ ft/s}^2}}$ represents the period of the pendulum. In this formula, $T$ is the period of the pendulum, in seconds, and $l$ is the length of the pendulum, in feet. Calculate the period of the pendulum. Express the answer to two decimal places.

**15.** A cone has a volume of 27 489 cm3 and a height of 14 cm. Using the formula $V = \frac{1}{3}\pi r2h$, determine the diameter of the cone. Express the answer to the nearest centimetre.

**16.** Chemical equilibrium applies to chemical reactions that can occur in two directions. When a chemical reaction reaches equilibrium, the rate of the forward reaction is equal to the rate of its reverse reaction. In the formula $Q = \frac{C}{A^2B^3}$, $Q$ represents the solutions at equilibrium, and $A$, $B$, and $C$ are three chemicals involved in a chemical reaction. Express each answer to two decimal places.

**a)** Determine the concentration of solution $B$ if solution $A$ has a concentration of 0.25 M, solution $C$ has a concentration of 0.12 M, and the value of $Q$ is 569.

**b)** Determine the concentration of solution $A$ if solution $B$ has a concentration of 0.32 M, solution $C$ has a concentration of 0.45 M, and the value of $Q$ is 26.

**17.** The national arena of Sweden, the Ericsson Globe, is considered the largest hemispherical building in the world. The interior of the Ericsson Globe has a volume of 696 910 $m^3$.

**a)** Determine the diameter of the arena using the formula $r = \sqrt[3]{\frac{3V}{4\pi}}$, where $r$ is the radius of the arena, in metres, and $V$ is the volume of the arena, in cubic metres. Express the answer to one decimal place.

**b)** Determine the surface area of the Ericsson Globe using the formula $SA = 4\pi r^2$, where $SA$ represents the surface area in square metres. Express the answer to one decimal place.

☆**18.** The surface area of a cylinder given its volume can be calculated using the formula $SA = 2\pi\left[h\left(\sqrt{\frac{V}{\pi h}}\right) + \left(\frac{V}{\pi h}\right)\right]$. Determine the surface area of a cylinder with height 26 m and volume of 26 465 $m^3$. Express the answer to the nearest square metre.

## C Extend

**19.** Without using a calculator, solve each of the following:

**a)** $\sqrt{\sqrt{16}}$

**b)** $\sqrt{\sqrt[3]{15\,625}}$

☆**c)** $\sqrt{4 + \sqrt{19 + \sqrt{36}}}$

☆**d)** $\sqrt[4]{13 + \sqrt[3]{22 + \sqrt[3]{125}}}$

**20.** Express as a power with a single rational exponent.

☆**a)** $\sqrt[3]{\sqrt{7}}$ ☆**b)** $\sqrt[4]{\sqrt[3]{5^2}}$

**c)** $\sqrt[5]{\sqrt{\frac{1}{8}}}$ **d)** $\sqrt[4]{\sqrt[3]{\left(\frac{2}{5}\right)^6}}$

## D Create Connections

**21.** Does the expression $\sqrt[4]{x^3}$ always have a solution? Explain your reasoning.

**22.** Does the expression $\sqrt[3]{x^4}$ always have a solution? Explain your reasoning.

**23.** Copy the table and add rows to fill in the information for the first 20 whole numbers.

| Number | Square Root | Answer |
|---|---|---|
| 1 | $\sqrt{1}$ | 1 |

Use your calculator to determine each square root and copy all of the digits showing on the calculator.

**a)** What do you notice about the square roots of numbers that are not perfect squares?

**b)** Are the square roots of all non-perfect squares irrational? Explain.